Mechanical Behavior of Arteries, Constitutive Modeling, and Parameter Estimation

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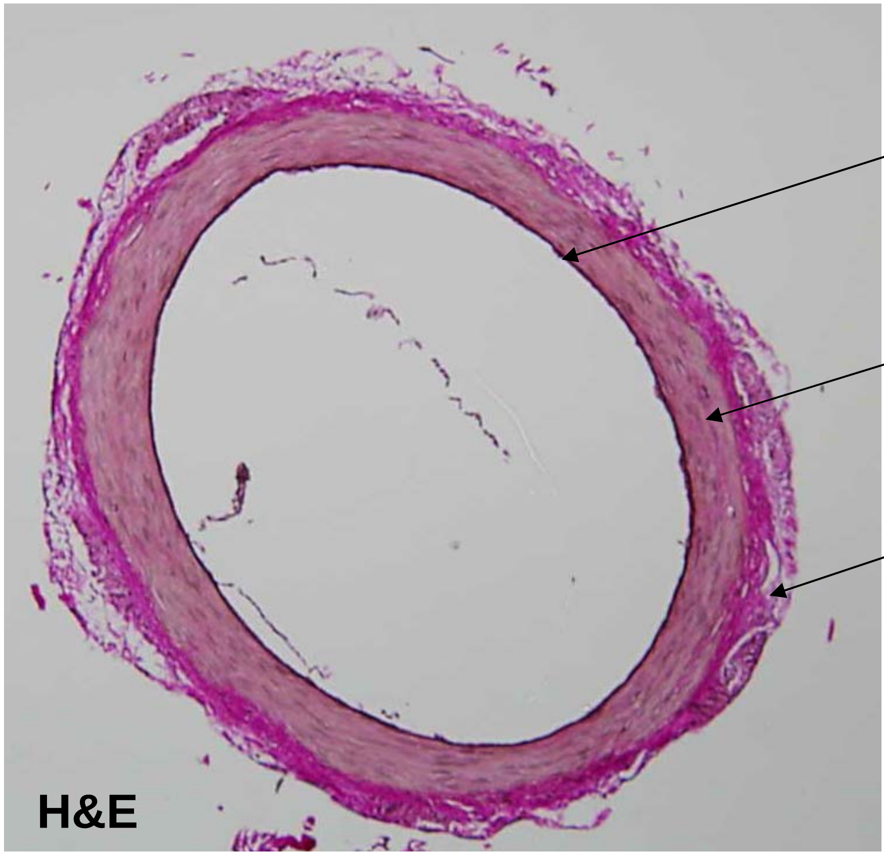
Contents

- Mechanical behavior of arteries
  - Histology
  - Experimental method
  - Constitutive modeling
- Parameter estimation and sensitivity coefficients
- Using residuals to improve parameter estimation
  - Weighted nonlinear least squares method
  - Choice of optimal number of parameters

The arterial wall



Cerebral Artery

Intima (inner layer)

- ECs, basement membrane

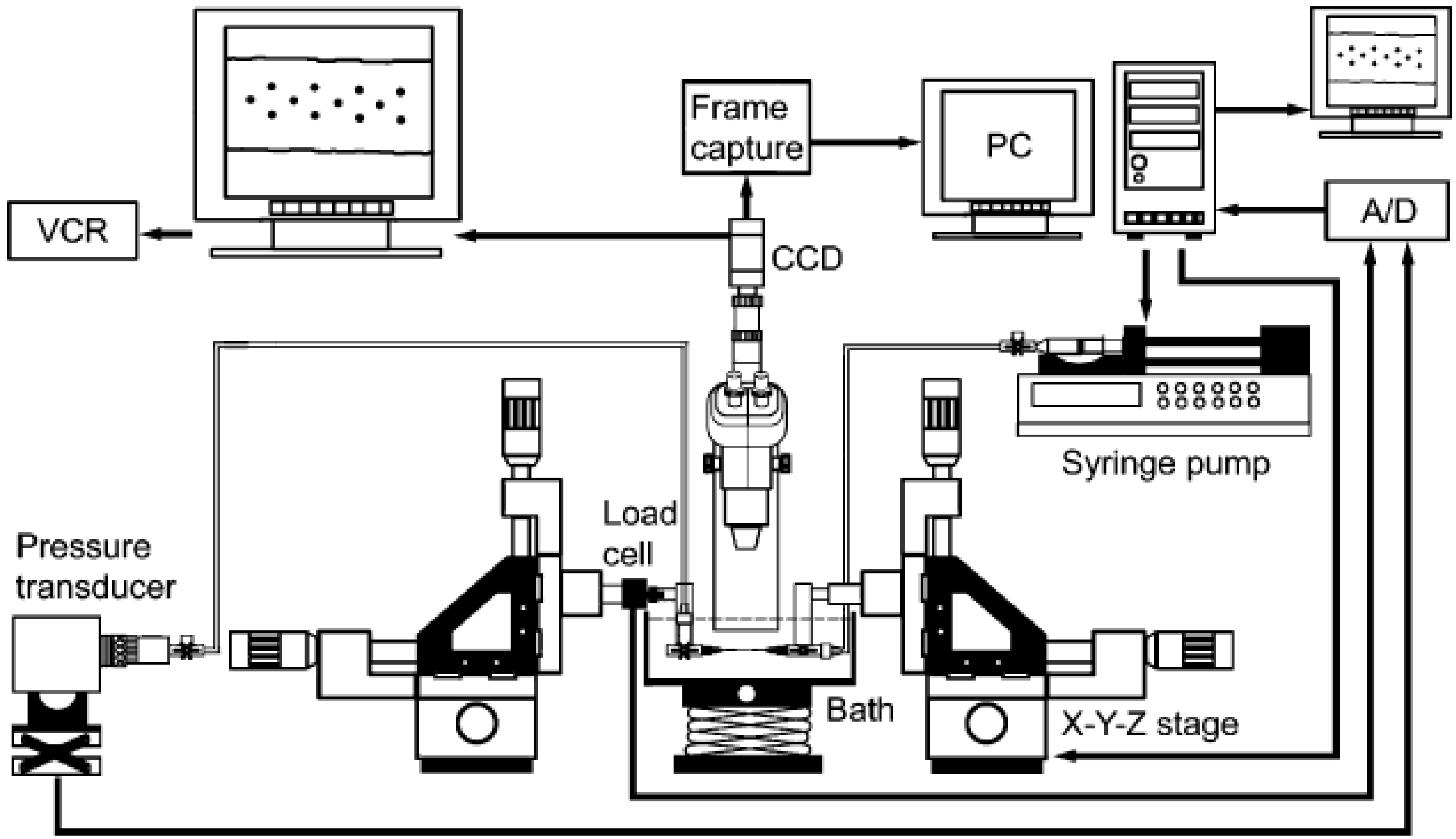
Media (middle layer)

- SMCs, collagen, elastin, PGs

Adventitia (outer layer)

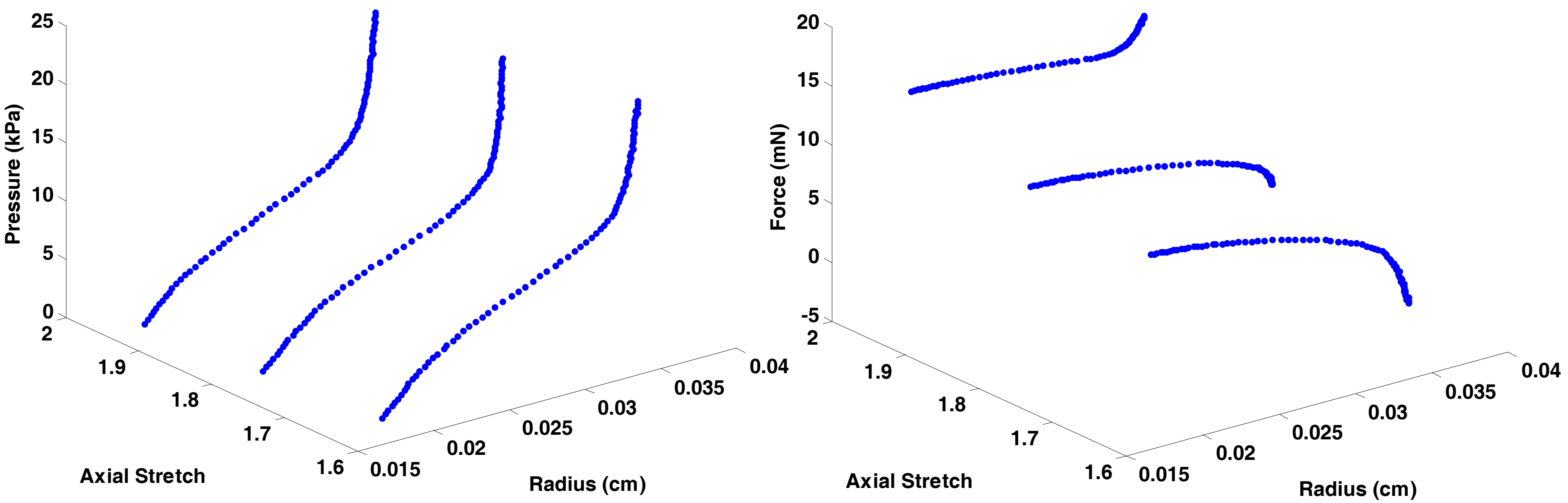
- FBs, collagen, vasa vasorum

Typical Experimental Set up for Artery Biaxial Test



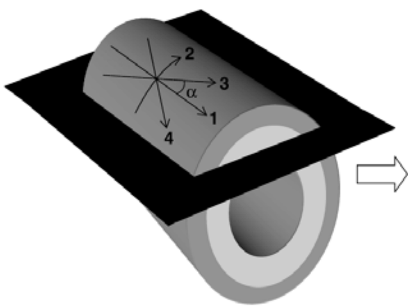
(Saravanan et al. 2006)

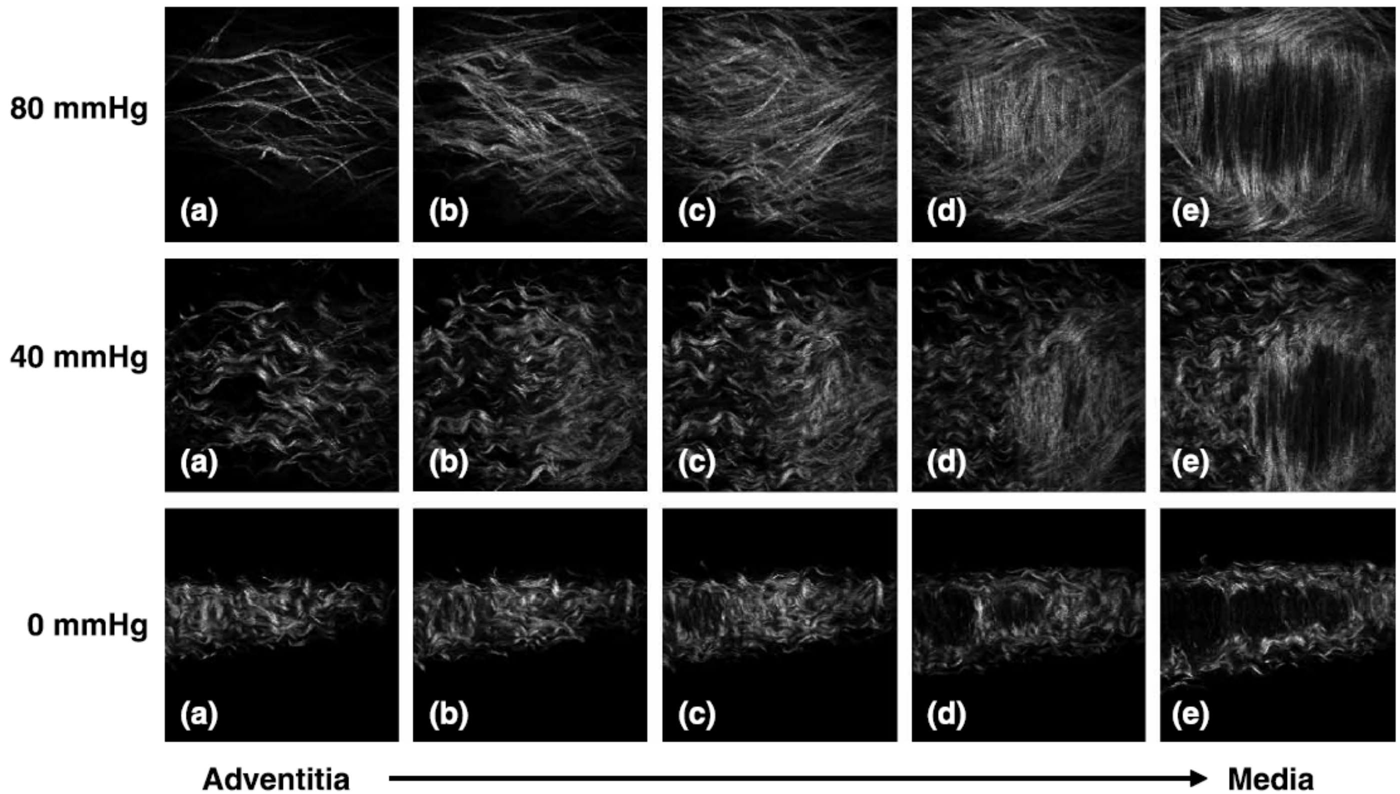
Experimental Data for Mouse Carotid Artery



Constitutive modeling: Multi-fiber family model

Images of collagen fibers during inflation test using nonlinear optical microscopy





(Wicker et al., 2009)

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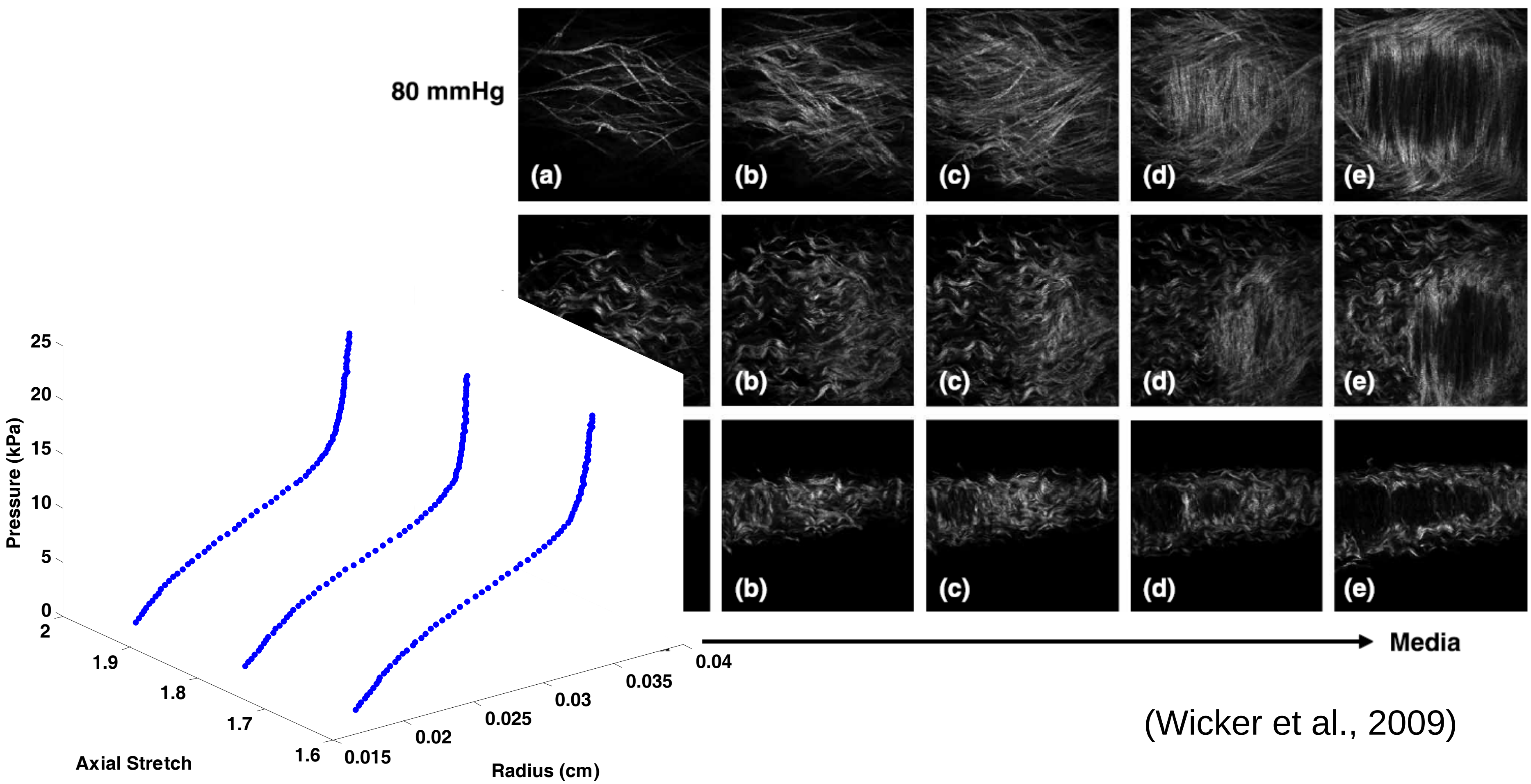

(Wicker et al., 2009)

Constitutive modeling

1. Force balance equation for a thin cylindrical vessel

$$P_i = \frac{h\left(\hat{T}_{\theta\theta} - \hat{T}_{rr}\right)}{r_m}$$

$$F_z = 2\pi r_m h\left(\hat{T}_{zz} - \hat{T}_{rr}\right) - \pi\left(r_m^2 - r_c^2\right) P_i$$

2. Stress for an incompressible hyperelastic material

$$\mathbf{T} = -p\mathbf{I} + \frac{2}{J}\mathbf{F}\frac{\partial \hat{W}}{\partial \mathbf{C}}\mathbf{F}^T$$

3. Strain energy function for arteries (Holzapfel et al., 2000; Baek et al., 2006)

$$\hat{W} = \frac{c}{2}(I_1 - 3) + \sum_k \frac{c_1^{(k)}}{4c_2^{(k)}}\left\{\exp\left(c_2^{(k)}(\lambda^{(k)^2} - 1)^2\right) - 1\right\},$$

where

$\lambda^{(k)} = \sqrt{\left(\lambda_z \sin\alpha^{(k)}\right)^2 + \left(\lambda_\theta \cos\alpha^{(k)}\right)^2}$ and $\lambda_\theta = r / R$

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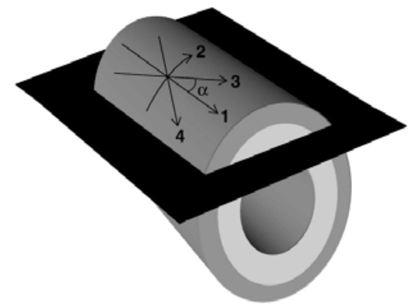

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Measurements

$$P_i, F_z, r, \lambda_z$$

Final constitutive equations

$$P_i = \hat{P}_i(r, \lambda_z)$$

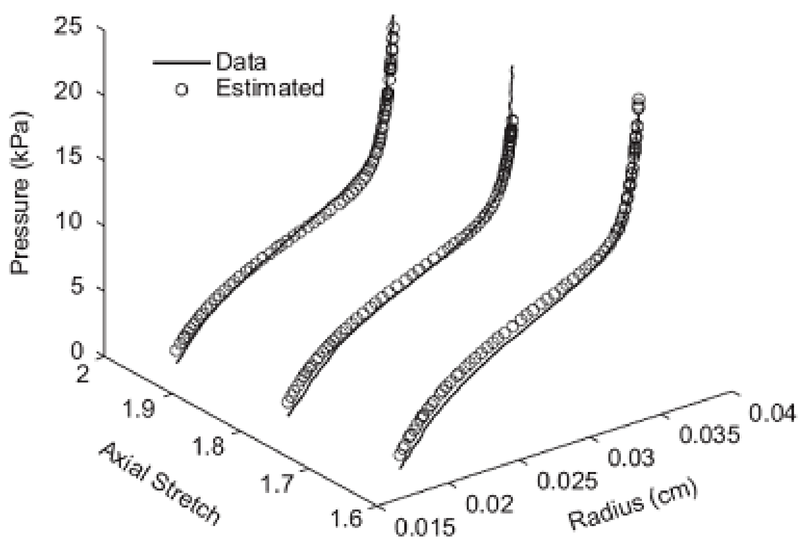
$$F_z = \hat{F}_z(r, \lambda_z)$$

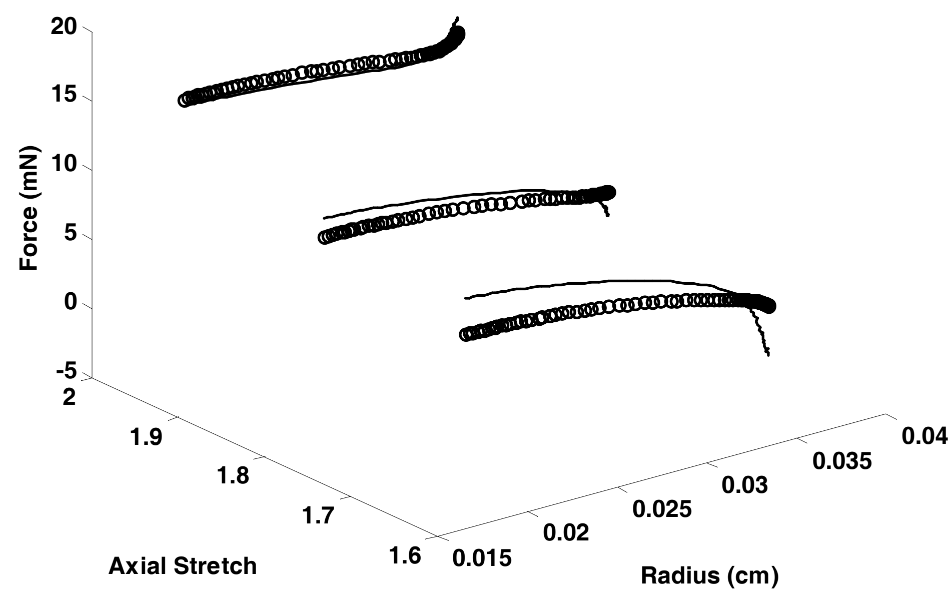
Parameter Estimation

Parameter Estimation for Mouse Carotid Artery

Nonlinear least squares method (NLSM) with 8 parameters

$$S = \sum_j \left(\frac{P_j^{data} - P_j^{calc}}{P_{invivo}} \right)^2 + \left(\frac{F_j^{data} - F_j^{calc}}{F_{invivo}} \right)^2$$

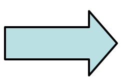


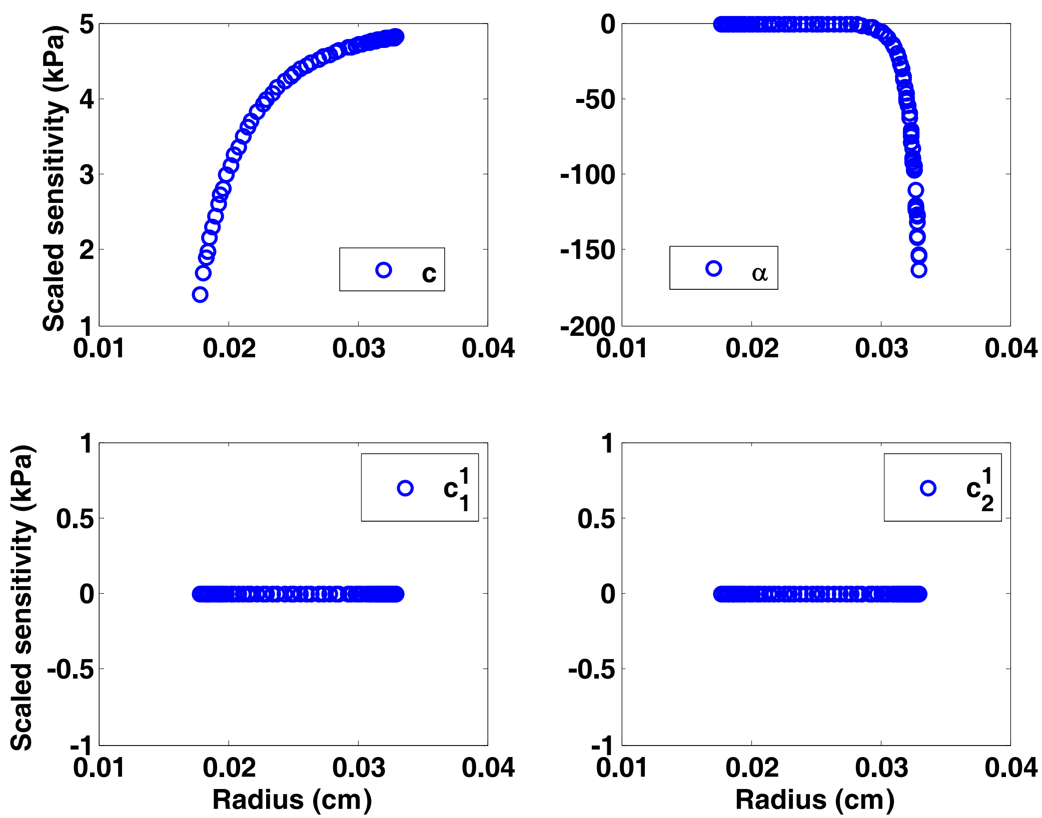


Sensitivity Coefficients for ***Pressure***

$$\hat{W} = \frac{c}{2}(I_1 - 3) + \sum_k \frac{c_1^{(k)}}{4c_2^{(k)}} \left\{ \exp\left(c_2^{(k)} (\lambda^{(k)^2} - 1)^2 \right) - 1 \right\}$$

$$\frac{\partial P_i}{\partial \theta}\theta$$





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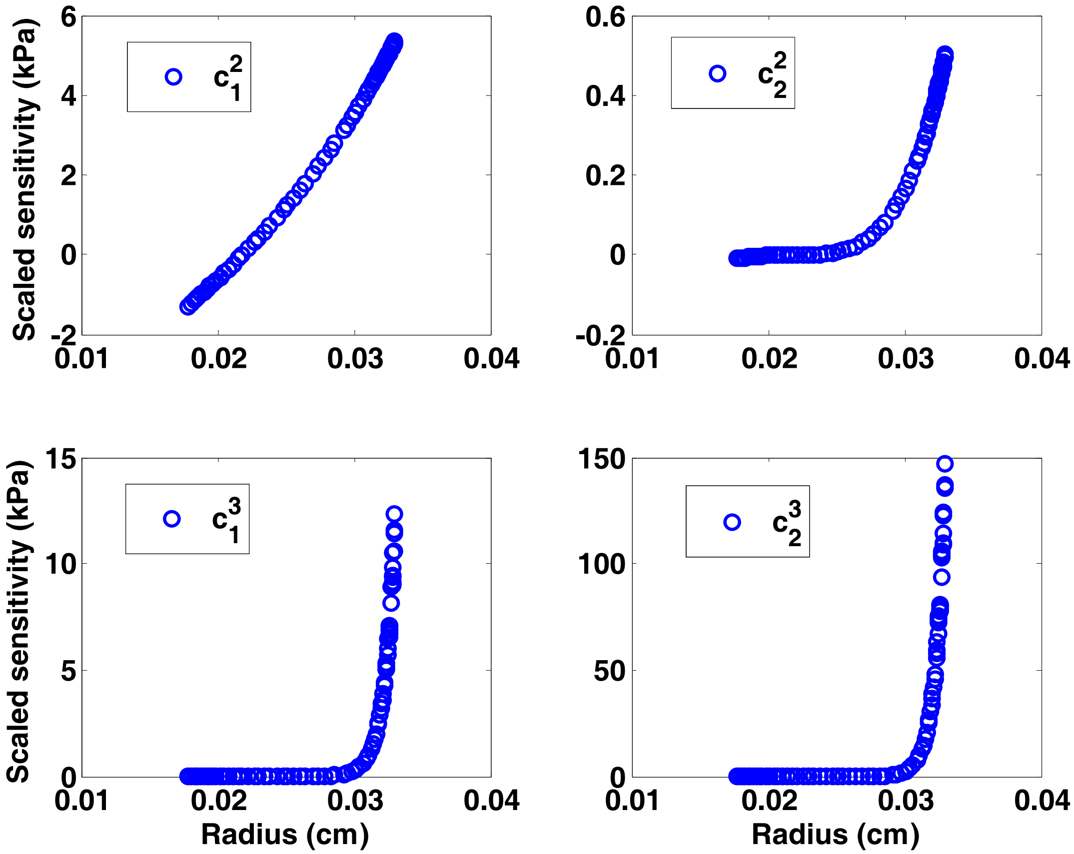

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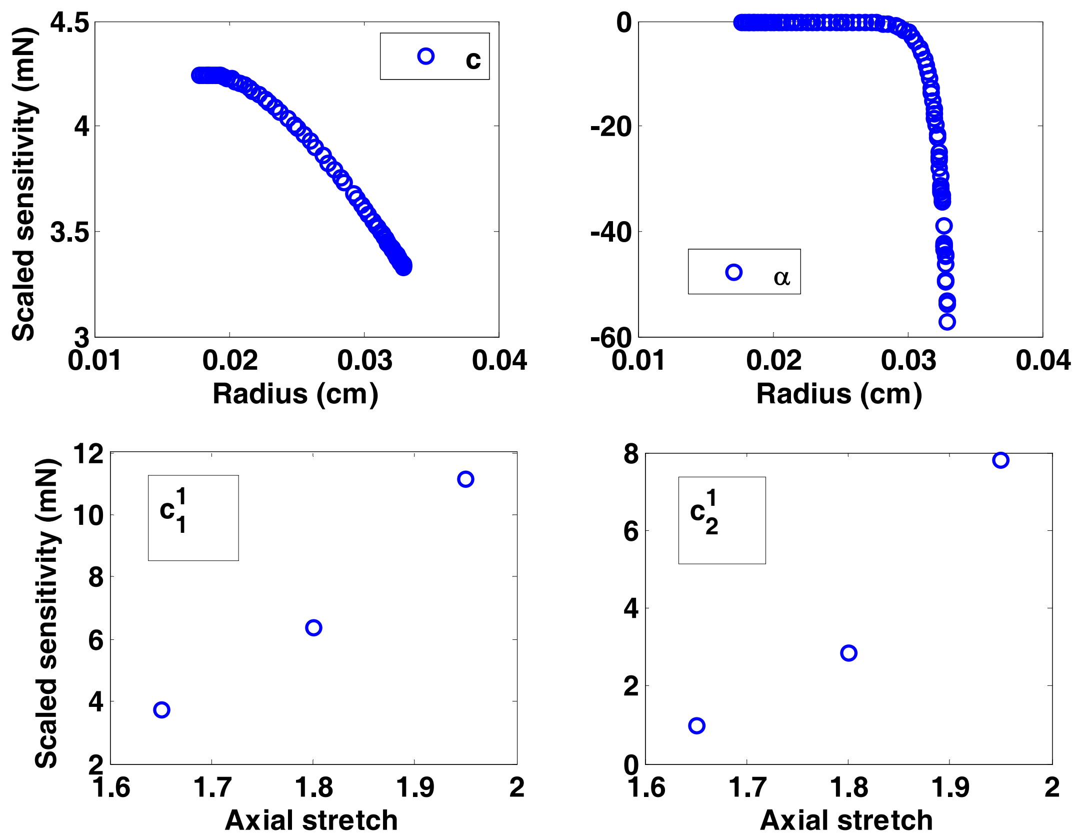
$$\frac{\partial F_z}{\partial \theta}\theta$$



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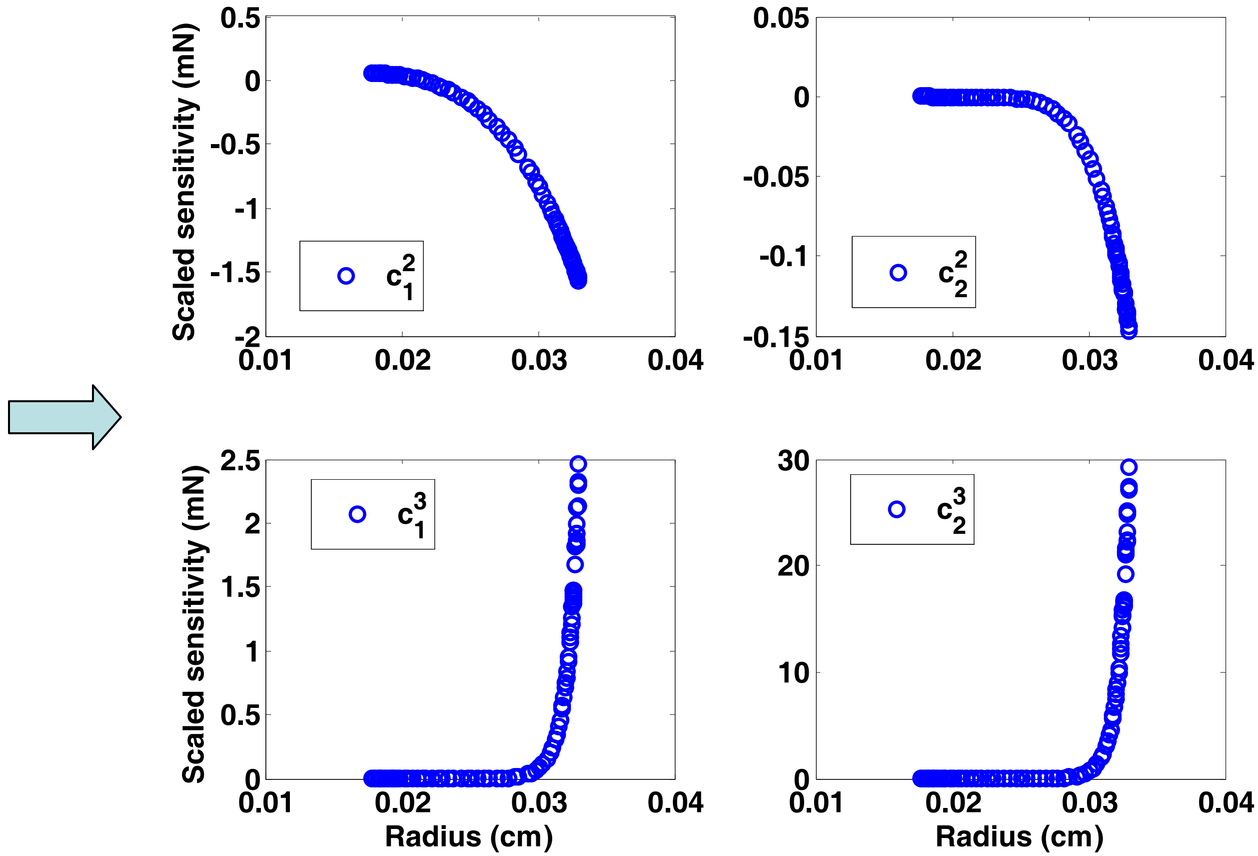
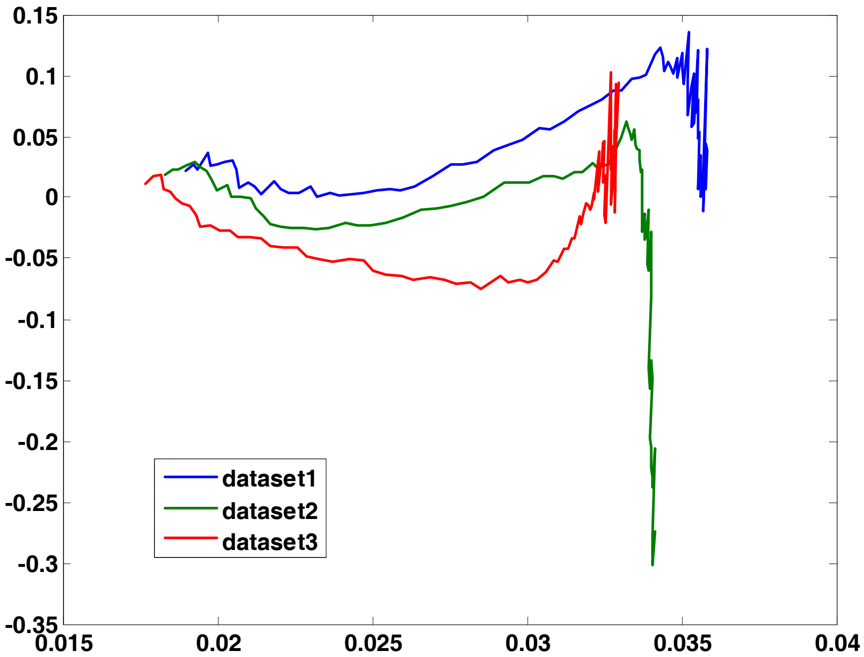


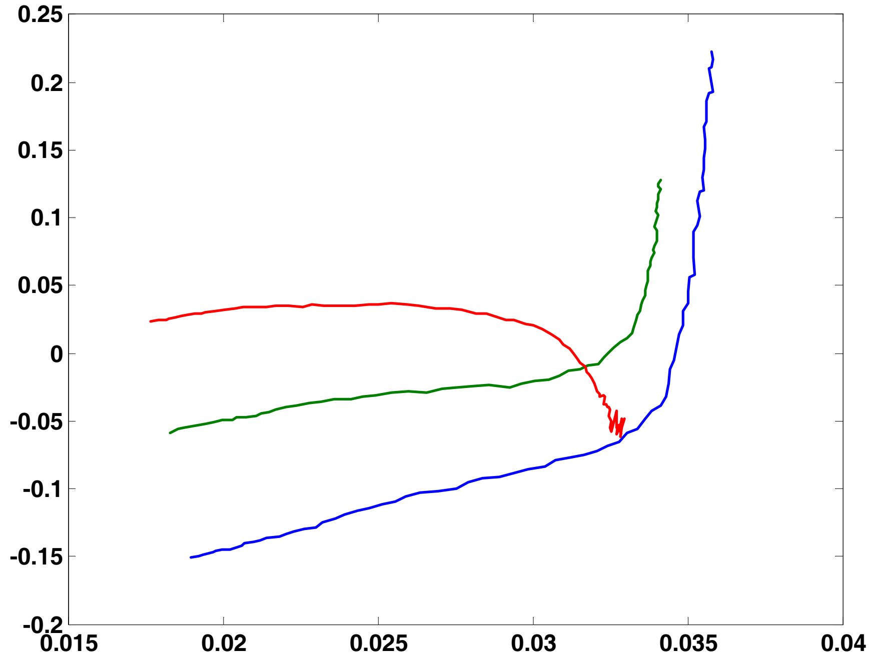
- Using residuals to improve parameter estimation
  - Noise model for weighted nonlinear least square method
  - Optimal number of parameters

Residuals: Pressure & Force Using NLS

$$R_p = \left(\frac{P^{data} - P^{calc}}{P_{invivo}} \right)$$

$$R_f = \left(\frac{F^{data} - F^{calc}}{F_{invivo}} \right)$$





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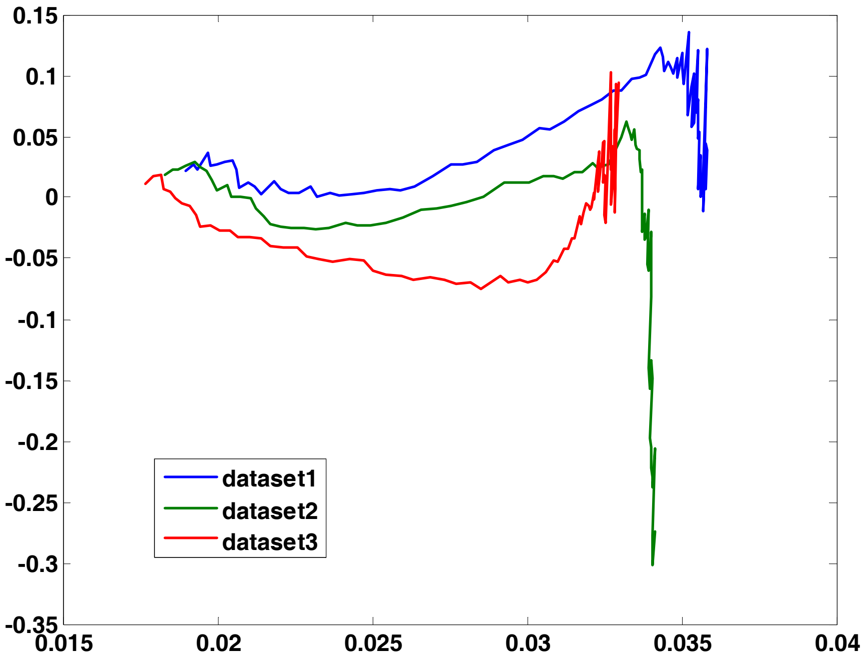
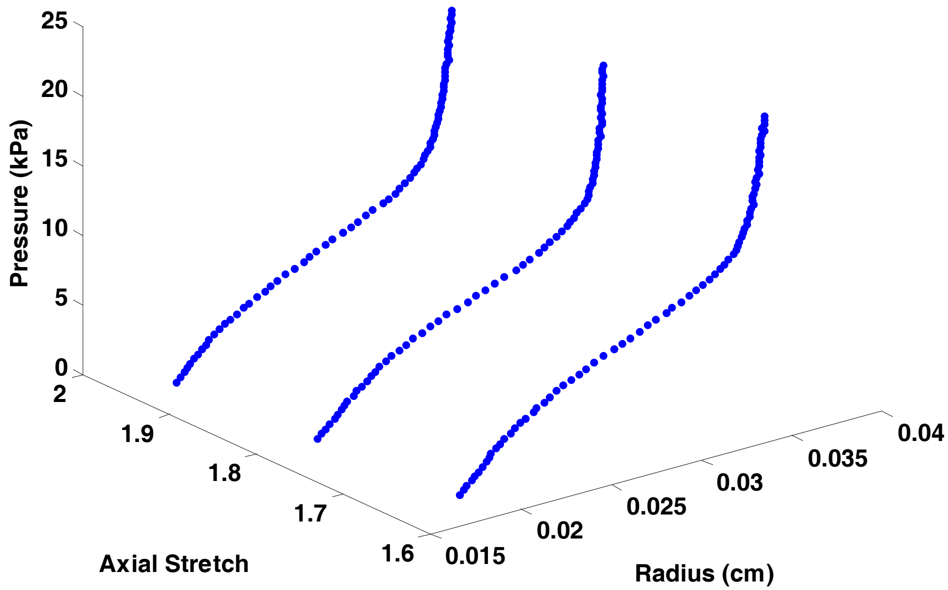




Measurement error in independent variables and noise propagation

1. Measurement error in variables

$$\hat{\mathbf{x}}_n = \tilde{\mathbf{x}}_n + \boldsymbol{\varepsilon}_n,$$

$$\hat{\mathbf{y}}_n = \tilde{\mathbf{y}}_n + \mathbf{e}_n,$$

2. Noise model $\mathbf{v}_n$ for $\hat{\mathbf{y}}_n - \mathbf{f}(\hat{\mathbf{x}}_n, \Theta)$

$$\mathbf{v}_n = -\frac{d\mathbf{f}}{d\mathbf{x}}(\tilde{\mathbf{x}}_n, \Theta)\boldsymbol{\varepsilon}_n + \mathbf{e}_n$$

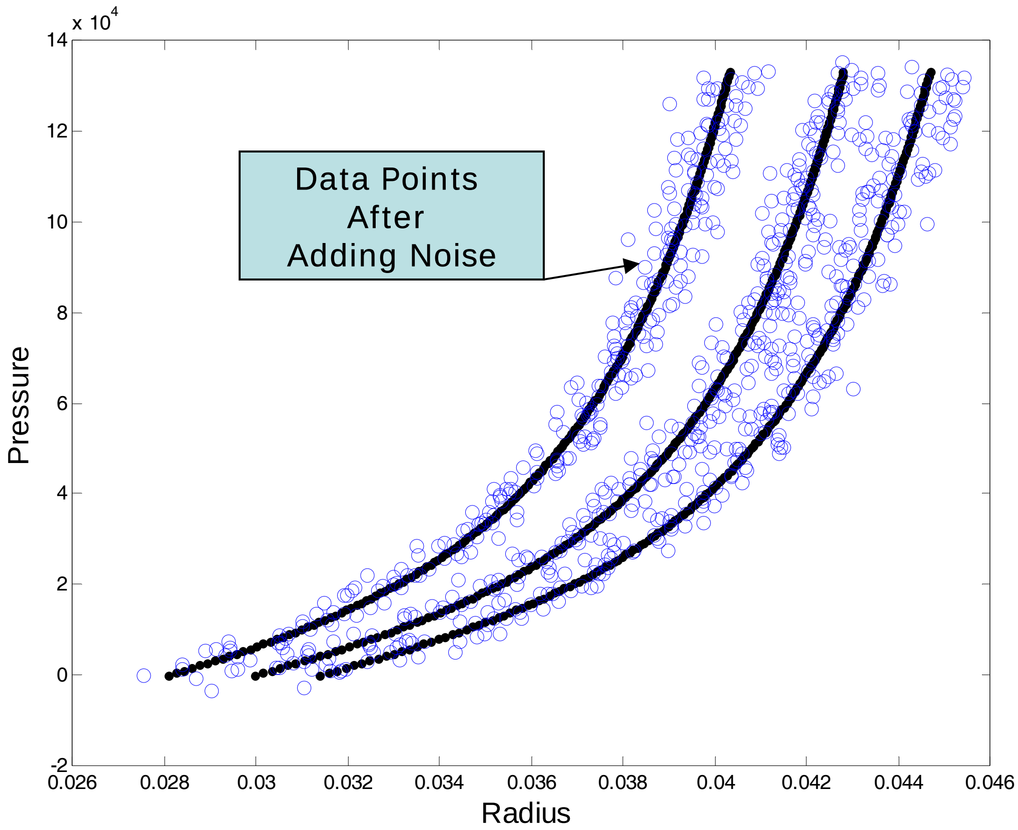
3. Weighted nonlinear least squares method

$$S = \sum_{n=1}^{m} (\hat{\mathbf{y}}_n - \mathbf{f}(\hat{\mathbf{x}}_n, \boldsymbol{\Theta}))^{\mathrm{T}} \boldsymbol{\Sigma}_{v_n}^{-1} (\hat{\mathbf{y}}_n - \mathbf{f}(\hat{\mathbf{x}}_n, \boldsymbol{\Theta}))$$

$$\boldsymbol{\Sigma}_{v_n} = \mathbb{E}(\mathbf{v}_n \mathbf{v}_n^{\mathrm{T}}) = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right) \boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}} \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)^{\mathrm{T}} + \boldsymbol{\Sigma}_{\mathbf{e}}$$

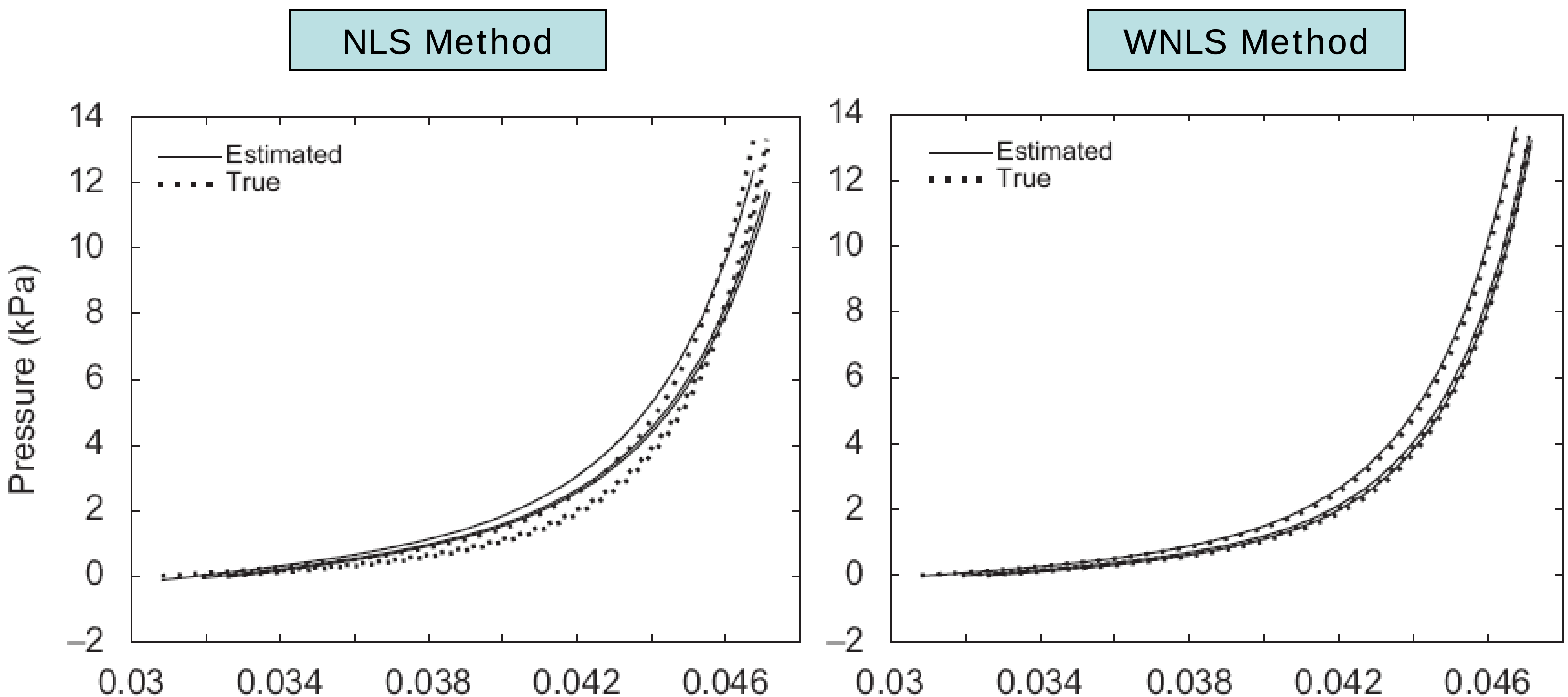
Parameter Estimation:
Validation: Synthesized Data

- Synthesized Noisy Data Generated
  Using White Gaussian Noise



Parameter Estimation: Validation: Synthesized Data

- Fitting results for pressure-radius



Determining Optimal Number of Parameters in the Model

Parameter Estimation: Optimal Number of Parameters/Fiber Families

- Going back to Strain Energy Function:

$$\hat{W} = \frac{c}{2}(I_1 - 3) + \sum_k \frac{c_1^{(k)}}{4c_2^{(k)}} \left\{ \exp\left(c_2^{(k)} ({\lambda^{(k)}}^2 - 1)^2 \right) - 1 \right\}$$

How many **fiber families** would be optimal?

- Increasing parameters **decreases the fitting residual**.

- But, it also increase the **complexity** of model;
  We may have **non-unique answers**

- There is a **tradeoff** between ***accuracy*** and ***complexity***

Parameter Estimation: Optimal Number of Fiber Families

- Model-selection criteria was tested

1. Akaike Information Criterion **(AIC)**

Optimization Residual

Number of Parameters

Number of Samples

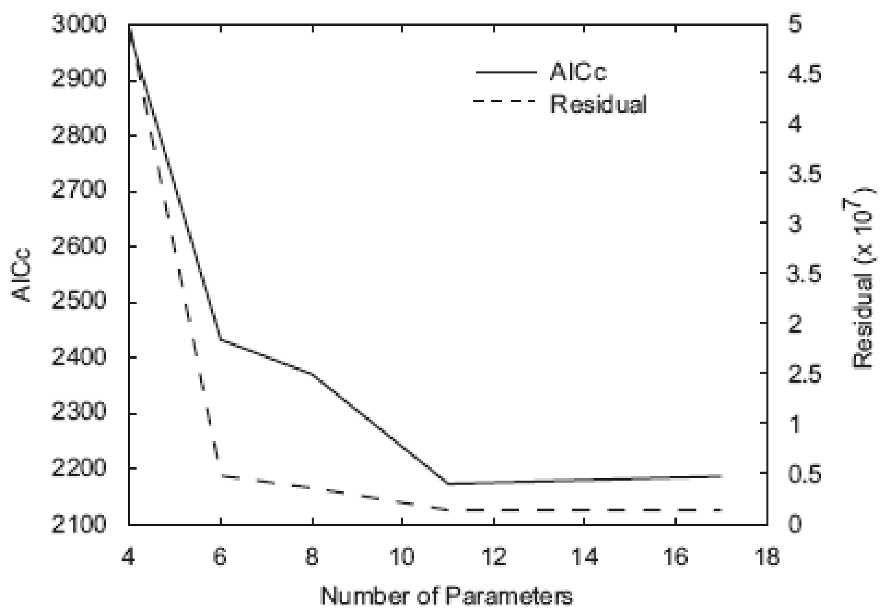
$$\mathrm{AIC} = m \ln\left(\frac{S}{m}\right) + 2(N+1),$$

2. A modified AIC

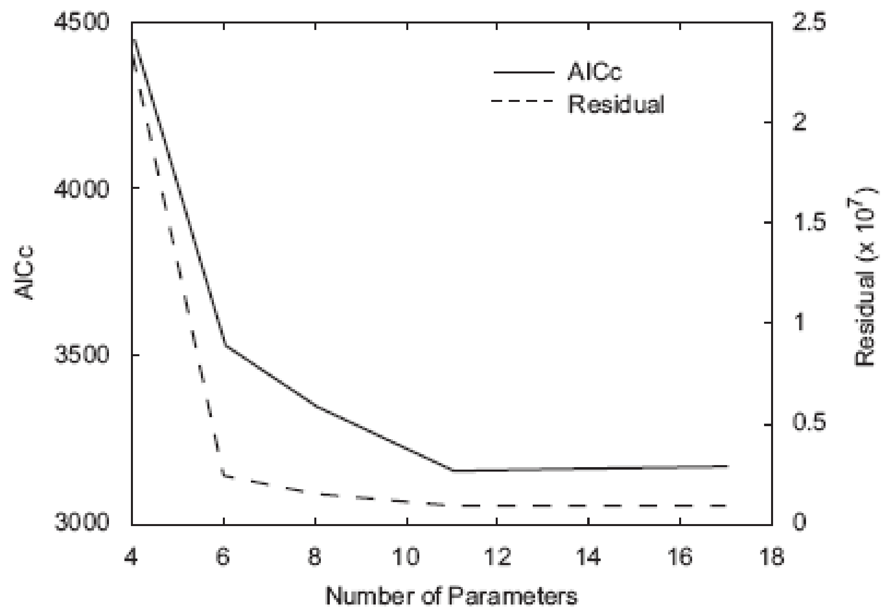
$$\mathrm{AICc} = \mathrm{AIC} + \frac{2(N+1)(N+2)}{m-N-2}$$

Parameter Estimation:
Optimal Number of Fiber Families

Mouse Carotid Artery



Rabbit Basilar Artery



Summary

1. This tutorial demonstrated how the sensitivity coefficients and residuals can be used to improve parameter estimation and in selection of number of parameter
2. Future work: developing new models and using information from residuals for model selection